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ABSTRACT

The record values of the Pareto distribution, labelled Pareto (II) (alpha, beta, nu), are reviewed. The best linear unbiased estimates of the parameters in terms of the record values are provided. The prediction of the sth record value based on the first m (s m) record values are obtained. A classical Pareto distribution provides reasonably good fits to distributions of income and of property values. Distributional properties of X sub L(n) and parameter estimations are illustrated using multiple equations. (Author/SLD)

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RECORD VALUES OF A PARETO DISTRIBUTION

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Abstract

This paper deals with the record values of the Pareto (II) distribution. Best linear unbiased estimates of the parameters in terms of the record values are given. The prediction of the sth record value based on the first m ($s > m$) record values are obatined.

Introduction

Suppose the random variable (rv) X has the probability density function (pdf) f(x) with the following cumulative distribution function F(x), where

$$F(x) = \left(1 + \frac{x-\alpha}{\beta}\right)^{-\nu}, \quad x \geq \alpha,\ \beta > 0 \text{ and } \nu > 0. \tag{1.1}$$

This is the classical Pareto distribution, labelled Pareto(II) (α, β, ν) by Arnold (1983).

We will say $X \in P_{\alpha,\beta,\nu}(x)$ if x has the F(x) as given by (1). It is well known that a Pareto distribution of the form as given in (1.1) provides reasonably good fits to distributions of income and of property values. For detailed discussion in this respect see Lomax (1954), Hagstroem (1960), Champernowne (1953) and Arnold (1983).

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distribution random variables with the cumulative distribution function (cdf) f(x) and the corresponding probability density function (pdf) f(x). Let $Y_n = \max(X_1, \ldots, X_n)$, for $n \geq 1$. We say X_j is a record value of $\{X_n\}$ if $Y_j > Y_{j-1}$. Define $L(1) = 1$ and $L(n) = \min \{j \mid j > L(n-1),\ X_j > X_{L(n-1)}\}$, $n > 1$. Then $X_{L(n)}$, $n \geq 1$ is the sequence of record values.

2. Distributional Properties of $X_{L(n)}$

We define the hazard rate r(x) by r(x) = (f(x)/(1-F(x)) and the cumulative hazard rate function R(x) by $R(x) = \int^{x} r(u)du$. For the rv with cdf as given in (1.1)

$$r(x) = \frac{\nu}{\beta}\left(1 + \frac{x-\alpha}{\beta}\right)^{-1}$$

The joint pdf f_1 of $X_{L(1)},\ldots, X_{L(n)}$ can be shown as (see Ahsanullah (1988))

(2.1) $\quad f_1(x_1,\ldots, x_n) = r(x_1)\ldots r(x_{n-1})\, f(x_n)$,

$$\text{for } \alpha < x_1 <\ldots< x_n < \infty$$

$$= 0, \quad \text{otherwise,}$$

Integrating out $x_1,\ldots, x_{m-1}, x_{m+1},\ldots, x_{n-1}$, we get the joint pdf f_2 of $X_{L(m)}$ and $X_{L(n)}$ as

(2.2) $\quad f_2(x_m x_n) = (\Gamma(n-m)\,\Gamma(m))^{-1} R^{m-1}(x_m)\, r(x_m)\, (R(x_n)-R(x_m))^{n-m-1}$

$$\text{for } \alpha < x_m < x_n < \infty$$

$$= 0, \quad \text{otherwise,}$$

where $R(x) = -\ln \bar{F}(x)$, for $\bar{F}(x) > 0$, $\bar{F}(x) = 1 - F(x)$.

Similarly it can be shown that pdf $f_{(m)}$ of $X_{L(m)}$ can be written as

(2.3) $\quad f_{(m)}(x_m) = (\Gamma(m+1))^{-1} R^{m}(x_m)\, f(x_m)$, for $\alpha < x_m < \infty$

$$= 0, \quad \text{otherwise.}$$

Hence the pdf f_3 of $X_{L(n)} | X_{L(m)} = x_m$ is

(2.4) $$f_3(x_n|X_{L(m)}=x_m) = (\lfloor^- (n-m))^{-1} (R(x_n) - R(x_m))^{n-m-1} f(x_n)(\bar{F}(x_m))^{-1}$$

$$\text{for } \alpha < x_m < x_n < \infty$$

$$= 0, \quad \text{otherwise.}$$

For record values of Pareto (I) distribution see Ahsanullah and Houchens (1989).

3. Estimation of the parameters

Let $\hat{\alpha}$ and $\hat{\beta}$ be the minimum variance unbiased estimates of α and β respectively based on the observed values $x_{L(1)}, \ldots, X_{L(m-1)}$, for known ν ($\nu>2$), then

(3.1) $$\hat{\alpha} = \frac{T(\nu-1)x_{L(1)}^{-B}}{\nu\ T-\nu+2}$$

(3.2) $$\hat{\beta} = \frac{T(\nu-1)x_{L(1)}^{-A+(\nu-2)B}}{\nu\ T-\nu+2}, \quad \text{where}$$

$$A = (\nu-1)(\nu-2)\ x_{L(1)} - \sum_{j=1}^{m} \left(\frac{\nu-2}{\nu}\right)^j x_{L(j)} - \left(\frac{\nu-2}{\nu}\right)^m (\nu-2)\ x_{L(m)}$$

$$B = \sum_{j=1}^{m} \left(\frac{\nu-2}{\nu}\right)^j x_{L(j)} + \left(\frac{\nu-2}{\nu}\right)^m (\nu-2)\ x_{L(m)}$$

$$T = \sum_{j=1}^{m} \left(\frac{\nu-2}{\nu}\right)^j .$$

$$\mathrm{Var}(\hat{\alpha}) = \beta^2 \frac{T}{(\nu-2)(\nu\ T-\nu+2)}$$

$$\mathrm{Var}(\hat{\beta}) = \beta^2 \frac{T+(\nu-2)^2}{(\nu-2)(\nu\ t-\nu+2}$$

$$\mathrm{Cov}(\hat{\alpha},\hat{\beta}) = \frac{\beta^2(T-\nu+2)}{(\nu-2)(\nu\ T-\nu+2)} \quad .$$

Proof:

We have

$$r(x) = \frac{\nu}{\beta}\left(1 + \frac{x-\alpha}{\beta}\right)^{-1}$$

and

$$R(x) = \nu \ln\left(1 + \frac{x-\alpha}{\beta}\right)$$

Substituting the values of r(x) and R(x) in (2.3) and (2.4), we get on simplification

$$E(X_{L(n)}) = \alpha - \beta + \beta \left(\frac{\nu}{\nu-1}\right)^n$$

$$\mathrm{Var}(X_{L(n)}) = \beta^2\left[\left(\frac{\nu}{\nu-2}\right)^n - \left(\frac{\nu}{\nu-1}\right)^{2n}\right]$$

$$\mathrm{Cov}(X_{L(m)},X_{L(n)}) = \left(\frac{\nu}{\nu-1}\right)^{n-m} V(X_{L(m-1)}),\ m \leq n.$$

We can write

$$\mathrm{Cov}(X_{L(m)},X_{L(n)}) = a_m\, b_n\, \beta^2,\ m \leq n\ ,$$

where

$$a_m = \left(\frac{\nu-1}{\nu}\right)^m \left[\left(\frac{\nu}{\nu-2}\right)^m - \left(\frac{\nu}{\nu-1}\right)^{2m}\right]$$

$$b_n = \left(\frac{\nu}{\nu-1}\right)^n$$

Let X be the m x 1 vector corresponding to the observed values of $X_{L(i)}$, i = 1,2,...,m.

Then we can write

$$E(X) = \alpha\, 1 + \beta\, \delta$$

$$Var(X) = \beta^2\, V$$

where $X' = (x_{L(1)}, \cdots, x_{L(m)})$

$1' = (1, 1, \ldots, 1)$

$$\delta' = \left(\left(\frac{\nu}{\nu-1}\right)-1,\ \left(\frac{\nu}{\nu-1}\right)^2-1,\ \ldots,\ \left(\frac{\nu}{\nu-1}\right)^m-1\right)$$

$V = (V_{ij})$, $V_{ij} = a_i b_j$, $1 \le i \le j \le m$.

Let $\Omega = V^{-1} = (V^{ij})$, then

$$V^{11} = \frac{a_2}{a_1(a_2b_1 - a_1b_2)}$$

$$V^{m\,m} = \frac{b_{m-1}}{b_m(a_m b_{m-1} - a_{m-1}b_m)}$$

$$V^{i+1,i} = V^{i,i+1} = -\frac{1}{a_{i+1}b_i - a_i b_{i+1}}\ ,\ i = 1,2,..,m-1$$

$$V^{ii} = \frac{a_{i+1}b_{i-1} - a_{i-1}b_{i+1}}{(a_i b_{i-1} - a_{i-1}b_i)(a_{i+1}b_i - a_i b_{i+1})}\ ,\ i = 2,\ldots,m-1$$

$$V^{ij} = 0 \quad \text{for } |i-j| \ge 2\ .$$

The minimum variance linear unbiased estimates $\hat{\alpha}$ and $\hat{\beta}$ of α and β respectively (see David (1970)) are

$$\hat{\alpha} = \delta' V^{-1} (1\,\delta' - \delta\,1') V^{-1} X/\Delta$$

$$\hat{\beta} = 1' V^{-1} (1\,\delta' - \delta\,1') V^{-1} X/\Delta$$

where

$$\Delta = (1' V^{-1} 1)(\delta' V^{-1} \delta) - (1' V^{-1} \delta)^2$$

and

$$\mathrm{Var}(\hat{\alpha}) = \beta^2\, 1' V^{-1} \delta/\Delta$$

$$\mathrm{Var}(\hat{\beta}) = \beta^2\, 1' V^{-1} 1/\delta$$

$$\mathrm{Cov}(\hat{\alpha},\hat{\beta}) = -\beta^2\, 1' V^{-1} \delta/\Delta\ .$$

On simplification it can be shown that

$$V^{ii} = (2\nu^2-4\nu+1)c^i\ ,\ i = 1,2,\ldots,m-1$$

$$V^{i+1,i} = -(\nu^2-3\nu+2)c^i = V^{i,i+1}\ ,\ i = 1,2,..,m-1$$

$$V^{m\,m} = (\nu^2-2\nu+1)c^m$$

$$V^{ij} = 0\ ,\ \text{if } |i-j| \geq 2$$

$$c = \nu/(\nu-2)$$

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\end{pmatrix} = \frac{1}{D}\begin{pmatrix} T & \nu-2+T \\ \nu-2+T & T+(\nu-2)^2\end{pmatrix}\begin{pmatrix}A\\ B\end{pmatrix}$$

$$D = (\nu-2)(\nu\,T-\nu+2)$$

$$T = \sum_{j=1}^{m} \left(\frac{\nu-2}{\nu}\right)^j$$

$$A = (\nu-1)(\nu-2)x_{L(1)} - \sum_{j=1}^{m} \left(\frac{\nu-2}{\nu}\right)^j x_{L(j)} - \left(\frac{\nu-2}{\nu}\right)^m (\nu-2)\, x_{L(m)}$$

$$B = \sum_{j=1}^{m} \left(\frac{\nu-2}{\nu}\right)^{j} x_{L(j)} + \left(\frac{\nu-2}{\nu}\right)^{m} (\nu-2)\, x_{L(m)} .$$

$$\hat{\alpha} = \frac{(\nu-1)\, T\, x_{L(1)} - B}{\nu\, T - \nu + 2}$$

$$\hat{\beta} = \frac{(\nu-1)\, T\, x_{L(1)} + (\nu-2)\, B - A}{\nu\, T - \nu + 2}$$

$$V(\hat{\alpha}) = \beta^2 \frac{T}{D}$$

$$V(\hat{\beta}) = \beta^2 \frac{T + (\nu-2)^2}{D}$$

$$\mathrm{Cov}(\hat{\alpha},\hat{\beta}) = \beta^2 \frac{(T-\nu+2)}{D}$$

4. Prediction of $X_{L(s)}$

We shall assume $s \geq m$. Let $W' = (W_1, W_2, \ldots, W_m)$, where $W = \mathrm{Cov}(X_{L(i)}, X_{L(s)})$, $i = 1,2,\ldots,m$ and $\sigma^* = \beta^{-1} E(X_{L(s)} - \alpha)$. It follows from the results of Goldberger [1962] that the best linear unbiased predictor of (BLUP) $X_{L(s)}$ is $\hat{X}_{L(s)}$, where

$$\hat{X}_{L(s)} = \hat{E}(X_{L(s)} + W'\, V^{-1}\, (X - \hat{\alpha}\, 1 - \hat{\beta}\, \delta) \tag{4.1}$$

with

$$\hat{E}(X_{L(s)} = \hat{\alpha} + \hat{\beta}\, \delta_s , \quad \delta_s = \left(\frac{\nu}{\nu-1}\right)^{s} - 1 .$$

Now

$$W'\, V^{-1} = (0,\ 0,\ \ldots,\ b_s/b_m)$$

$$W'\, V^{-1}(X - \hat{\alpha} - \hat{\beta}\, \delta) = \frac{b_s}{b_m} (x_{L(m)} - \hat{\alpha} - \hat{\beta}\, \delta_m), \quad \delta_m = \left(\frac{\nu}{\nu-1}\right)^{m} - 1$$

Hence

$$\hat{X}_{L(s)} = \left(\frac{\nu}{\nu-1}\right)^{s-m} x_{L(m)} + \left(1 - \left(\frac{\nu}{\nu-1}\right)^{s-m}\right)(\hat{\alpha} - \hat{\beta}) .$$

Let $\tilde{X}_{L(s)}$ be the best invariant predictor (BLIP) of $X_{L(s)}$. From the results of Mann (1969) if follows that

$$\tilde{X}_{L(s)} = \hat{X}_{L(s)} - (C_{12}(1+C_{22})^{-1})\,\hat{\beta} , \tag{4.2}$$

where

$$C_{12}\,\beta^2 = \mathrm{Cov}(\hat{\beta} , (1-W' V^{-1} 1)\,\hat{\alpha} + (\delta^*-W' V^{-1} \delta)\hat{\beta})$$

$$C_{22}\,\beta^2 = \mathrm{Var}\,\hat{\beta} ,$$

Now

$$C_{12} = \left(1 - \frac{b_s}{b_m}\right)\frac{T-\nu+2}{D} + \left(\left(\frac{\nu}{\nu-1}\right)^{s} - 1 - \left(\frac{\nu}{\nu-1}\right)^{s-m}\right)\left(\left(\frac{\nu}{\nu-1}\right)^{m} - 1\right)\frac{T+(\nu-2)^2}{D}$$

$$= \left(\left(\frac{\nu}{\nu-1}\right)^{s-m} - 1\right)\frac{(\nu-2)(\nu-1)}{D}$$

$$C_{12}\,(1+C_{22})^{-1} = \left(\left(\frac{\nu}{\nu-1}\right)^{s-m} - 1\right)\frac{\nu-2}{T(\nu-1)} .$$

Thus the BLIP of $X_{L(s)}$ is

$$\tilde{X}_{L(s)} = \hat{X}_{L(s)} - \left(\left(\frac{\nu}{\nu-1}\right)^{s-m} - 1\right)\frac{\nu-2}{T(\nu-1)}\,\hat{\beta} \tag{4.3}$$

$$E(\tilde{X}_{L(s)} = E(\hat{X}_{L(s)} - \left(\left(\frac{\nu}{\nu-1}\right)^{s-m} -1\right)\frac{\nu-2}{T(\nu-1)}\,\beta$$

$$= \alpha + d\,\beta ,$$

where

$$d = \left(\frac{\nu}{\nu-1}\right)^{s} - 1 - \left(\left(\frac{\nu}{\nu-1}\right)^{s-m} - 1\right)\frac{\nu-2}{T(\nu-1)} .$$

$$\text{MSE}\ (\tilde{X}_{L(s)}) = E(\tilde{X}_{L(s)} - X_{L(s)})^2$$

$$= \text{MSE}\ (\hat{X}_{L(s)}) - C_2\ \beta^2\ ,$$

where $C_2 = C_{12}^2/(1 + C_{22})$.

It is well known that the best (unrestricted) least squares predictor $\bar{X}_{L(s)}$ of $X_{L(s)}$ is

$$\bar{X}_{L(s)} = E\ (X_{L(s)} | X_{L(1)}, X_{L(2)}, \ldots, X_{L(m)})\ . \tag{4.4}$$

By using (2.4), we have

$$\bar{X}_{L(s)} = \left(\frac{n}{\nu-1}\right)^{s-m} x_{L(m)} + \left(1 - \left(\frac{\nu}{\nu-1}\right)^{s-m}\right)(\alpha-\beta)\ . \tag{4.5}$$

But $\bar{X}_{L(s)}$ depends on the unknown paramters α and β. If we substitute the minimum variance unbiased estimates of α and β in (3.5), then $\bar{X}_{L(s)}$ becomes equal to $\hat{X}_{L(s)}$.

By considering the mean square errors of $\hat{X}_{L(s)}$, $\tilde{X}_{L(s)}$ and $\bar{X}_{L(s)}$, we see that

$$\text{MSE}\ (\bar{X}_{L(s)}) < \text{MSE}\ (\tilde{X}_{L(s)}) < \text{MSE}\ (\hat{X}_{L(s)}).$$

For $m = 4$ and $\nu = 2.5$, we have

$$\hat{\alpha} = 1.4064\ x_{L(1)} - .3226\ x_{L(2)} - .0654\ x_{L(3)} - .0194\ x_{L(4)}$$

and

$$\hat{\beta} = -.6097\ x_{L(1)} + .4839\ x_{L(2)} + .0968\ x_{L(3)} + .0290\ x_{L(4)}$$

The variances and covariances of $\hat{\alpha}$ and $\hat{\beta}$ are given by

$$\text{Var}(\hat{\alpha}) = 4.0258\ \beta^2$$

$$\text{Var}(\hat{\beta}) = 8.0581\ \beta^2$$

$$\text{Cov}\ (\hat{\alpha},\hat{\beta}) = -\ 4.0387\ \beta^2$$

The predicted values $\hat{X}_{L(5)}$ and $\tilde{X}_{L(5)}$ of $X_{L(5)}$ are

$$\hat{X}_{L(5)} = \frac{5}{3}\ X_{L(4)} - \frac{2}{3}\ (\hat{\alpha} - \hat{\beta})$$

and

$$\tilde{X}_{L(5)} = \hat{X}_{L(5)} - .8903\ \hat{\beta}\ .$$

$$\text{MSE}\ (\tilde{X}_{L(5)}) < \text{MSE}\ (\tilde{X}_{L(5)})$$

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